

On multivalent solutions of Riemann-Hilbert problem

VLADIMIR RYAZANOV

Abstract - It is proved the existence of multivalent solutions for the Riemann-Hilbert problem in the general settings of finitely connected domains bounded by mutually disjoint Jordan curves, measurable coefficients and measurable boundary data. The theorem is formulated in terms of harmonic measure and principal asymptotic values. It is also given the corresponding reinforced criterion for domains with rectifiable boundaries stated in terms of the natural parameter and nontangential limits. Furthermore, it is shown that the dimension of the spaces of these solutions is infinite.

Key words and phrases : Riemann-Hilbert problem, Jordan curves, harmonic measures, principal asymptotic values, nontangential limits.

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1 Introduction

This note is a continuation of the paper [10] where the Riemann-Hilbert problem was resolved in these general settings for simply connected domains and where you could find the brief history of the question and the corresponding references, see also [11], [12] and [13]. Here, on the basis of [10] and a theorem due to Poincare, see e.g. Section VI.1 in [4], it is given a resolution of the problem for finitely connected domains.

2 The case of circular domains

Let us start from the simplest kind of multiply connected domains. Recall that a domain D in $\overline{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$ is called **circular** if its boundary consists of a finite number of mutually disjoint circles and points. We call such a domain **nondegenerate** if its boundary consists only of circles.

Theorem 2.1 *Let \mathbb{D}_* be a bounded nondegenerate circular domain and let $\lambda : \partial\mathbb{D}_* \rightarrow \mathbb{C}$, $|\lambda(\zeta)| \equiv 1$, and $\varphi : \partial\mathbb{D}_* \rightarrow \mathbb{R}$ be measurable functions. Then there exist multivalent analytic functions $f : \mathbb{D}_* \rightarrow \mathbb{C}$ such that*

$$\lim_{z \rightarrow \zeta} \operatorname{Re} \{ \overline{\lambda(\zeta)} \cdot f(z) \} = \varphi(\zeta) \quad (1)$$

along any nontangential path to a.e. $\zeta \in \partial\mathbb{D}_$.*

Proof. Indeed, by the Poincare theorem, see e.g. Theorem VI.1 in [4], there is a locally conformal mapping g of the unit disc $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ onto \mathbb{D}_* . Let $h : \mathbb{D}_* \rightarrow \mathbb{D}$ be the corresponding multivalent analytic function that is inverse to g . \mathbb{D}_* without a finite number of cuts is simply connected and hence h has there only single-valued branches that are extended to the boundary by the Caratheodory theorem.

By Section VI.2 in [4], $\partial\mathbb{D}$ except a countable set of its points consists of a countable collection of arcs every of which is a one-to-one image of a circle in $\partial\mathbb{D}_*$ without its one point under an extended branch of h . Note that by the reflection principle g is conformally extended into a neighborhood of every such arc and, thus, nontangential paths to its points go into nontangential paths to the corresponding points of circles in $\partial\mathbb{D}_*$ and inversely.

Setting $\Lambda = \lambda \circ g$ and $\Phi = \varphi \circ g$ with the extended g on the given arcs of $\partial\mathbb{D}$ we obtain the suitable measurable functions on $\partial\mathbb{D}$. By Theorem 2.1 in [10] or in [12] there exist analytic functions $F : \mathbb{D} \rightarrow \mathbb{C}$ such that

$$\lim_{w \rightarrow \eta} \operatorname{Re} \{ \overline{\Lambda(\eta)} \cdot F(w) \} = \Phi(\eta) \quad (2)$$

along any nontangential path to a.e. $\eta \in \partial\mathbb{D}$. By the above arguments $f = F \circ h$ are desired multivalent analytic solutions of (1). \square

In particular, choosing $\lambda \equiv 1$ in (1), we obtain the following statement.

Proposition 2.1 *Let \mathbb{D}_* be a bounded nondegenerate circular domain and let $\varphi : \partial\mathbb{D}_* \rightarrow \mathbb{R}$ be a measurable function. Then there exist multivalent analytic functions $f : \mathbb{D}_* \rightarrow \mathbb{C}$ such that*

$$\lim_{z \rightarrow \zeta} \operatorname{Re} f(z) = \varphi(\zeta) \quad (3)$$

along any nontangential path to a.e. $\zeta \in \partial\mathbb{D}_$.*

3 Domains bounded by rectifiable Jordan curves

To resolve the Riemann-Hilbert problem in the case of domains bounded by a finite number of Jordan curves we should extend to the case the known results of Caratheodory (1912), Lindelöf (1917), F. and M. Riesz (1916) and Lavrentiev (1936).

Lemma 3.1 *Let D be a bounded domain in \mathbb{C} whose boundary consists of a finite number of mutually disjoint Jordan curves, \mathbb{D}_* be a bounded nondegenerate circular domain in \mathbb{C} and let $\omega : D \rightarrow \mathbb{D}_*$ be a conformal mapping. Then*

- (i) ω can be extended to a homeomorphism of \overline{D} onto $\overline{\mathbb{D}_*}$;
- (ii) $\arg [\omega(\zeta) - \omega(z)] - \arg [\zeta - z] \rightarrow \text{const}$ as $z \rightarrow \zeta$ whenever ∂D has a tangent at $\zeta \in \partial D$;
- (iii) for rectifiable ∂D , $\text{length } \omega^{-1}(E) = 0$ whenever $|E| = 0$, $E \subset \partial \mathbb{D}_*$;
- (iv) for rectifiable ∂D , $|\omega(\mathcal{E})| = 0$ whenever $\text{length } \mathcal{E} = 0$, $\mathcal{E} \subset \partial D$.

Proof. (i) Indeed, we are able to transform \mathbb{D}_* into a simply connected domain \mathbb{D}^* through a finite sequence of cuts. Thus, we come to the desired conclusion applying the Caratheodory theorems to simply connected domains \mathbb{D}^* and $D^* := \omega^{-1}(\mathbb{D}^*)$, see e.g. Theorem 9.4 in [2] and Theorem II.C.1 in [5].

(ii) In the construction from the previous item, we may assume that the point ζ is not the end of the cuts in D generated by the cuts in \mathbb{D}_* under the extended mapping ω^{-1} . Thus, we obtain to the desired conclusion twice applying the Caratheodory theorems, the reflection principle for conformal mappings and the Lindelöf theorem for the Jordan domains, see e.g. Theorem II.C.2 in [5].

Points (iii) and (iv) are proved similarly to the last item on the basis of the corresponding results of F. and M. Riesz and Lavrentiev for Jordan domains with rectifiable boundaries, see e.g. Theorem II.D.2 in [5], and [6], see also the point III.1.5 in [8]. \square

Theorem 3.1 *Let D be a bounded domain in \mathbb{C} whose boundary consists of a finite number of mutually disjoint rectifiable Jordan curves and $\lambda : \partial D \rightarrow \mathbb{C}$, $|\lambda(\zeta)| \equiv 1$, and $\varphi : \partial D \rightarrow \mathbb{R}$ be measurable functions with respect to the natural parameter on ∂D . Then there exist multivalent analytic functions $f : \mathbb{D} \rightarrow \mathbb{C}$ such that along any nontangential path*

$$\lim_{z \rightarrow \zeta} \text{Re} \{ \overline{\lambda(\zeta)} \cdot f(z) \} = \varphi(\zeta) \quad \text{for a.e. } \zeta \in \partial D \quad (4)$$

with respect to the natural parameters of the boundary components of D .

Proof. This case is reduced to the case of a bounded nondegenerate circular domain \mathbb{D}_* in the following way. First, there is a conformal mapping ω of D onto a circular domain \mathbb{D}_* , see e.g. Theorem V.6.2 in [4]. Note that \mathbb{D}_* is not degenerate because isolated singularities of conformal mappings are removable that is due to the well-known Weierstrass theorem, see e.g. Theorem 1.2 in [2]. Applying in the case of need the inversion with respect to a boundary circle of \mathbb{D}_* , we may assume that \mathbb{D}_* is bounded.

By point (i) in Lemma 3.1 ω can be extended to a homeomorphism of \overline{D} onto $\overline{\mathbb{D}_*}$. If ∂D is rectifiable, then by point (iii) in Lemma 3.1 $\text{length } \omega^{-1}(E) = 0$ whenever $E \subset \partial \mathbb{D}_*$ with $|E| = 0$, and by (iv) in Lemma 3.1, conversely, $|\omega(\mathcal{E})| = 0$ whenever $\mathcal{E} \subset \partial D$ with $\text{length } \mathcal{E} = 0$.

In the last case ω and ω^{-1} transform measurable sets into measurable sets. Indeed, every measurable set is the union of a sigma-compact set and a set of measure zero, see e.g. Theorem III(6.6) in [14], and continuous mappings transform compact sets into compact sets. Thus, a function $\varphi : \partial D \rightarrow \mathbb{R}$ is measurable with respect to the natural parameter on ∂D if and only if the function $\Phi = \varphi \circ \omega^{-1} : \partial \mathbb{D}_* \rightarrow \mathbb{R}$ is measurable with respect to the linear measure on $\partial \mathbb{D}_*$.

By point (ii) in Lemma 3.1, if ∂D has a tangent at a point $\zeta \in \partial D$, then $\arg [\omega(\zeta) - \omega(z)] - \arg [\zeta - z] \rightarrow \text{const}$ as $z \rightarrow \zeta$. In other words, the conformal images of sectors in D with a vertex at ζ is asymptotically the same as sectors in \mathbb{D}_* with a vertex at $w = \omega(\zeta)$. Thus, nontangential paths in D are transformed under ω into nontangential paths in \mathbb{D}_* and inversely. Finally, a rectifiable Jordan curve has a tangent a.e. with respect to the natural parameter and, thus, Theorem 3.1 follows from Theorem 2.1. \square

In particular, choosing $\lambda \equiv 1$ in (4), we obtain the following statement.

Proposition 3.1 *Let D be a bounded domain in \mathbb{C} whose boundary consists of a finite number of mutually disjoint rectifiable Jordan curves and let $\varphi : \partial D \rightarrow \mathbb{R}$ be measurable. Then there exist multivalent analytic functions $f : D \rightarrow \mathbb{C}$ such that*

$$\lim_{z \rightarrow \zeta} \text{Re } f(z) = \varphi(\zeta) \quad \text{for a.e. } \zeta \in \partial D \quad (5)$$

along any nontangential path with respect to the natural parameters of the boundary components of ∂D .

4 Domains bounded by arbitrary Jordan curves

See detailed comments on terms of **harmonic measures** and **principal asymptotic values** in the paper [10] or in [12].

Theorem 4.1 *Let D be a bounded domain in \mathbb{C} whose boundary consists of a finite number of mutually disjoint Jordan curves and let $\lambda : \partial D \rightarrow \mathbb{C}$, $|\lambda(\zeta)| \equiv 1$, and $\varphi : \partial D \rightarrow \mathbb{R}$ be measurable functions with respect to harmonic measures in D . Then there exist multivalent analytic functions $f : \mathbb{D} \rightarrow \mathbb{C}$ such that*

$$\lim_{z \rightarrow \zeta} \operatorname{Re} \{ \overline{\lambda(\zeta)} \cdot f(z) \} = \varphi(\zeta) \quad \text{for a.e. } \zeta \in \partial D \quad (6)$$

with respect to harmonic measures in D in the sense of the unique principal asymptotic value.

Proof. By the reasons of the first item in the proof of Theorem 3.1, there is a conformal mapping ω of D onto a bounded nondegenerate circular domain \mathbb{D}_* in \mathbb{C} . Set $\Lambda = \lambda \circ \Omega$ and $\Phi = \varphi \circ \Omega$ where $\Omega := \omega^{-1}$ extended to $\partial \mathbb{D}_*$ by point (i) in Lemma 3.1.

Note that harmonic measure zero is invariant under conformal mappings. Thus, arguing as in the third item of the proof to Theorem 3.1, we conclude that the functions Λ and Φ are measurable with respect to harmonic measures in \mathbb{D}_* .

By Theorem 2.1 there exist multivalent analytic functions $F : \mathbb{D}_* \rightarrow \mathbb{C}$ such that

$$\lim_{w \rightarrow \eta} \operatorname{Re} \{ \overline{\Lambda(\eta)} \cdot F(w) \} = \Phi(\eta)$$

along any nontangential path to a.e. $\eta \in \partial \mathbb{D}_*$.

By the construction the functions $f := F \circ \omega$ are desired multivalent analytic solutions of (6) in view of the Bagemihl theorem, see e.g. Theorem 2 in [1] or Theorem III.1.8 in [7]. \square

In particular, choosing $\lambda \equiv 1$ in (6), we obtain the following consequence.

Proposition 4.1 *Let D be a bounded domain in \mathbb{C} whose boundary consists of a finite number of mutually disjoint Jordan curves and let $\varphi : \partial D \rightarrow \mathbb{R}$ be a measurable function with respect to harmonic measures in D . Then there exist multivalent analytic functions $f : D \rightarrow \mathbb{C}$ such that*

$$\lim_{z \rightarrow \zeta} \operatorname{Re} f(z) = \varphi(\zeta) \quad \text{for a.e. } \zeta \in \partial D \quad (7)$$

with respect to harmonic measures in D in the sense of the unique principal asymptotic value.

5 On the dimension of spaces of solutions

By the Lindelöf maximum principle, see e.g. Lemma 1.1 in [3], it follows the uniqueness theorem for the Dirichlet problem in the class of bounded harmonic functions on the unit disk. Our multivalent analytic solutions are generally speaking not bounded and we have the new phenomena.

Theorem 5.1 *The spaces of multivalent analytic solutions in Theorems 2.1, 3.1 and 4.1 and in Propositions 2.1, 3.1 and 4.1 have the infinite dimension.*

Proof. By Theorem 5.1 in [10] or in [12] the space of solutions of the problem (2) has the infinite infinite dimension. Thus, the conclusion follows by the construction of solutions in the given theorems. \square

References

- [1] F. Bagemihl, Curvilinear cluster sets of arbitrary functions, *Proc. Nat. Acad. Sci. U.S.A.*, **41** (1955), 379-382.
- [2] E. F. Collingwood, A. J. Lohwater, *The theory of cluster sets*, Cambridge Tracts in Math. and Math. Physics, No. 56, Cambridge Univ. Press, Cambridge, 1966.
- [3] Garnett J.B., Marshall D.E., *Harmonic Measure*, Cambridge Univ. Press, Cambridge, 2005.
- [4] G. M. Goluzin, *Geometric theory of functions of a complex variable*, Transl. of Math. Monographs, Vol. 26, American Mathematical Society, Providence, R.I. 1969.
- [5] P. Koosis, *Introduction to H_p spaces*, 2nd ed., Cambridge Tracts in Mathematics, 115, Cambridge Univ. Press, Cambridge, 1998.
- [6] M. Lavrentiev, On some boundary problems in the theory of univalent functions, *Mat. Sbornik N.S.* **1**(43), 6 (1936), 815846 [in Russian].
- [7] Noshiro K. *Cluster sets*, Springer-Verlag, Berlin etc., 1960.
- [8] I. I. Priwalow, *Randeigenschaften analytischer Funktionen*, Hochschulbücher für Mathematik, Bd. 25, Deutscher Verlag der Wissenschaften, Berlin, 1956.
- [9] M. Riesz, Sur les fonctions conjuguées, *Math. Z.* **27**, no. 2 (1927), 218-244.
- [10] V. Ryazanov, *On the Riemann-Hilbert Problem without Index*, *Ann. Univ. Bucharest, Ser. Math.* **5 (LXIII)**, no. 1 (2014), 169-178.
- [11] V. Ryazanov, *Infinite dimension of solutions of the Dirichlet problem*, *Open Math.* (the former *Central European J. Math.*) **13**, no. 1 (2015), 348-350.
- [12] V. Ryazanov, *On the Riemann-Hilbert problem IV* // arXiv: 1308.2486v10 [math.CV] 11 Feb 2014, 1-10.
- [13] V. Ryazanov, *Infinite dimension of solutions for the Dirichlet problem* // arXiv: 1402.2130v5 [math.CV] 30 Jul 2014, 1-5.
- [14] S. Saks, *Theory of the integral*, Warsaw, 1937; Dover Publications Inc., New York, 1964.

Vladimir Ryazanov

Institute of Applied Mathematics and Mechanics

National Academy of Sciences of Ukraine

1 Dobrovol'skogo Str., 84100 Slavyansk, Ukraine

E-mail: vl.ryazanov1@gmail.com